

RICCI-QUADRATIC HOMOGENEOUS RANDERS SPACES*

SHAOQIANG DENG AND ZHIGUANG HU

Abstract

A Finsler space is called Ricci-quadratic if its Ricci curvature $Ric(x, y)$ is quadratic in y . It is called a Berwald space if its Chern connection defines a linear connection directly on the underlying manifold M . In this article, we prove that a homogeneous Randers space is Ricci-quadratic if and only if it is of Berwald type.

Mathematics Subject Classification(2000):22F30,53B20,53C60.

Key words: homogeneous Randers spaces; Ricci-quadratic metric; Berwald metric

1. Introduction

Riemann curvature is a central concept in Riemannian geometry which was introduced by Riemann in 1854. In 1926, Berwald generalized this notion to Finsler metrics. A Finsler metric is said to be R-quadratic if its Riemann curvature is quadratic [5]. R-quadratic metrics were first introduced by Basco and Matsumoto [2]. They form a rich class of Finsler spaces. For example, all Berwald metrics are R-quadratic, and some non-Berwald R-quadratic Finsler metrics have been constructed in [3, 9]. There are many interesting works related to this subject (cf. [12, 10]).

Ricci curvature of a Finsler space is the trace of the Riemann curvature. A Finsler metric is called Ricci-quadratic if its Ricci curvature $Ric(x, y)$ is quadratic in y . It is clear that the notion of Ricci-quadratic metrics is weaker than that of R-quadratic metrics. It is therefore obvious that any R-quadratic Finsler space must be Ricci-quadratic, in particular, any Berwald space must be Ricci-quadratic. However, there are many non-Berwald spaces which are Ricci-quadratic. In general, it is quite difficult to characterize Ricci-quadratic metrics. Li and Shen considered the case of Randers metrics in [9] and obtained a characterization of Ricci-quadratic properties of such spaces, using some complicated calculations in local coordinate systems. Their results are rather complicated (see Theorem 3.1 below). In this paper we consider homogeneous Randers spaces and prove the following

Main theorem. *A homogeneous Randers space is Ricci-quadratic if and only if it is of Berwald type.*

We remark here that the range of homogeneous Randers spaces is rather wide. For example, on any connected Lie group G with Lie algebra \mathfrak{g} , we can identify the tangent space of G at the origin $T_o(G)$ with \mathfrak{g} . Given any inner product $\langle \cdot, \cdot \rangle$ and a vector $w \in \mathfrak{g}$ with $\langle w, w \rangle < 1$, we can then define a Minkowski norm F_o on \mathfrak{g} by (see [1])

$$F_o(u) = \sqrt{\langle u, u \rangle} + \langle w, u \rangle.$$

Then we can extend this Minkowski norm to a left invariant Randers metric F on G by the left translation of G . This method produces numerous examples of homogeneous Randers spaces. It is also known that on many coset spaces G/H of a Lie group G with respect to a non-trivial closed subgroup H , there exist G -invariant Randers metrics (see for example [6]).

*Project supported by NSFC (no. 10971104, 10671096) and SRFDP of China.

Zhiguang Hu is the corresponding author. E-mail address: nankaitaiji@mail.nankai.edu.cn.

2. Preliminaries

Let F be a Finsler metric on an n -dimensional manifold M . We always assume that F is positive definite, namely, the Hessian matrix $g_{ij} = g_{ij}(x, y)$ is positive definite, where

$$g_{ij}(x, y) := \frac{1}{2}[F^2]_{y^i y^j}(x, y), \quad y \in T_x M - \{0\}.$$

On a standard local coordinate system $(x^1, x^2, \dots, x^n, y^1, y^2, \dots, y^n)$, the geodesics of F are characterized by the following system of equations:

$$\frac{d^2 x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0,$$

where $G^i = G^i(x, y)$ are called the geodesic coefficients of F , which are given by

$$G^i = \frac{1}{4} \left\{ [F^2]_{x^m y^i} y^m - [F^2]_{x^i} \right\}.$$

It is clear that if F is Riemannian, then G^i are quadratic in y . For a general Finsler metric, G^i is very complicated and is not quadratic in y . When G^i are quadratic in y , we call F a Berwald metric. Every Riemannian metric is a Berwald metric, but the converse is not true. In fact, one can construct many examples of non-Riemannian Berwald metrics. The local structure of Berwald spaces was determined by Z. I. Szabó in [13].

A Finsler metric of the form $F = \alpha + \beta$, where α is a Riemannian metric and β is a 1-form on M whose length with respect to α is everywhere less than 1, is called a Randers metric. This kind of metrics was introduced by G. Randers in 1941 ([11]), in his study of general relativity. A Randers metric $F = \alpha + \beta$ is a Berwald metric if and only if the form β is parallel with respect to α . This is an important result in the field of Finsler geometry due to the contributions of many mathematicians, see [1] for an account of the history of this result.

Let y be a non-zero vector in $T_x(M)$. The Riemann curvature $R_y = R^i_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i_k := 2[G^i]_{x^k} - [G^i]_{x^m y^k} y^m + 2G^m [G^i]_{y^m y^k} - [G^i]_{y^m} [G^m]_{y^k}.$$

It defines a linear transformation on $T_x(M)$. The trace of this linear transformation is denoted by $Ric(x, y)$ and is called the Ricci curvature of F . A Finsler metric is called Ricci quadratic if $Ric(x, y)$ is quadratic in y .

3. THE LEVI-CIVITA CONNECTION OF HOMOGENEOUS SPACES

In this section we shall use Killing vector fields to present some formulas about the Levi-Civita connection of homogeneous Riemannian manifolds. We follow the method used by the first author in [6].

Let $(G/H, \alpha)$ be a homogeneous Riemannian manifold. Then the Lie algebra of G has a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where \mathfrak{h} is the Lie algebra of H and $Ad(h)(\mathfrak{m}) \subset \mathfrak{m}, \forall h \in H$. We identify \mathfrak{m} with the tangent space $T_o(G/H)$ of the origin $o = H$. We shall use the notation \langle, \rangle to denote the Riemannian metric on the manifold as well as its restriction to \mathfrak{m} . Note that it is an AdH -invariant inner product on \mathfrak{m} . Hence we have

$$\langle [x, u], v \rangle + \langle [x, v], u \rangle = 0, \quad \forall x \in \mathfrak{h}, \forall u, v \in \mathfrak{m},$$

which is equivalent to

$$\langle [x, u], u \rangle = 0, \quad \forall x \in \mathfrak{h}, \forall u \in \mathfrak{m}.$$

Given $v \in \mathfrak{g}$, we can define the fundamental vector field \hat{v} generated by v , i.e.,

$$\hat{v}_{gH} = \frac{d}{dt} \exp(tv) gH|_{t=0}, \quad \forall g \in G.$$

Since the one-parameter transformation group $\exp tv$ on G/H consists of isometries, \hat{v} is a Killing vector field.

Let $\hat{X}, \hat{Y}, \hat{Z}$ be Killing vector fields on G/H and U, V be arbitrary smooth vector fields on G/H . Then we have ([4], page 40,182,183)

$$[\hat{X}, \hat{Y}] = -[X, Y], \quad (3.1)$$

$$\hat{X}\langle U, V \rangle = \langle [\hat{X}, U], V \rangle + \langle [\hat{X}, V], U \rangle, \quad (3.2)$$

$$\langle \nabla_{\hat{X}} \hat{Y}, \hat{Z} \rangle = -\frac{1}{2} \left(\langle [X, Y], \hat{Z} \rangle + \langle [X, Z], \hat{Y} \rangle + \langle [Y, Z], \hat{X} \rangle \right). \quad (3.3)$$

We only need to prove (2.2). In fact, by (c) of Theorem 1.81 of [4], we have:

$$\begin{aligned} \hat{X}\langle U, V \rangle &= \langle \nabla_{\hat{X}} U, V \rangle + \langle U, \nabla_{\hat{X}} V \rangle \\ &= \langle \nabla_U \hat{X}, V \rangle + \langle [\hat{X}, U], V \rangle + \langle U, \nabla_V \hat{X} \rangle + \langle U, [\hat{X}, V] \rangle \\ &= \langle [\hat{X}, U], V \rangle + \langle [\hat{X}, V], U \rangle. \end{aligned}$$

Let u_1, u_2, \dots, u_n be an orthonormal basis of \mathfrak{m} with respect to $\langle \cdot, \cdot \rangle$. We extend it to a basis u_1, u_2, \dots, u_m of \mathfrak{g} . By [8], there exists a local coordinate system on a neighborhood V of o , which is defined by the mapping

$$(\exp(x^1 u_1) \exp(x^2 u_2) \cdots \exp(x^n u_n))H \rightarrow (x^1, x^2, \dots, x^n).$$

Let $gH = (x^1, x^2, \dots, x^n) \in U$. Then

$$\left. \frac{\partial}{\partial x^i} \right|_{gH} = f_i^a \hat{u}_a|_{gH},$$

where

$$f_i^a u_a = e^{x^1 a d u_1} \cdots e^{x^{i-1} a d u_{i-1}}(u_i). \quad (3.4)$$

Remark In the following, the indices a, b, c, \dots range from 1 to m , the indices i, j, k, \dots range from 1 to n and the indices λ, μ, \dots range from $n+1$ to m .

Let Γ_{ij}^l be the Christoffel symbols in the coordinate system, i.e.,

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

Then

$$\Gamma_{ij}^l \frac{\partial}{\partial x^l} = \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \frac{\partial f_j^a}{\partial x^i} \hat{u}_a + f_i^b f_j^a \nabla_{\hat{u}_b} \hat{u}_a. \quad (3.5)$$

From (3.4), we see that f_i^a are functions of x^1, \dots, x^{i-1} . Thus

$$\frac{\partial f_j^a}{\partial x^i} = 0, \quad i \geq j.$$

Therefore (3.5) gives

$$\Gamma_{ij}^l \frac{\partial}{\partial x^l} = f_i^b f_j^a \nabla_{\hat{u}_b} \hat{u}_a, \quad i \geq j.$$

Differentiating the above equation with respect to x_k , we get

$$\frac{\partial \Gamma_{ij}^l}{\partial x^k} \frac{\partial}{\partial x^l} + \Gamma_{ij}^s \Gamma_{ks}^l \frac{\partial}{\partial x^l} = \frac{\partial (f_i^b f_j^a)}{\partial x^k} \nabla_{\hat{u}_b} \hat{u}_a + f_i^b f_j^a f_k^c \nabla_{\hat{u}_c} \nabla_{\hat{u}_b} \hat{u}_a, \quad i \geq j. \quad (3.6)$$

Differentiating (3.4) with respect to x_k and letting $(x^1, \dots, x^n) \rightarrow 0$, we obtain

$$\frac{\partial f_i^a}{\partial x^k}(0) = f(k, i) C_{ki}^a,$$

where C_{ab}^c are the structure constants of \mathfrak{g} which are defined by $[u_a, u_b] = C_{ab}^c u_c$ and $f(k, l)$ are defined by

$$f(k, i) := \begin{cases} 1, & k < i, \\ 0 & k \geq i. \end{cases}$$

Considering the value at the origin o we get

Lemma 3.1.

$$\Gamma_{ij}^l(o) = f(i, j)C_{ij}^l + \langle \nabla_{\hat{u}_i} \hat{u}_j, \hat{u}_l \rangle, \quad (3.7)$$

$$\begin{aligned} \left. \frac{\partial \Gamma_{ij}^l}{\partial x^k} \right|_o &= -\Gamma_{ij}^s(\Gamma_{ks}^l + \langle \nabla_{\hat{u}_k} \hat{u}_l, \hat{u}_s \rangle) + f(k, j)C_{kj}^a \langle \nabla_{\hat{u}_i} \hat{u}_a, \hat{u}_l \rangle \\ &\quad + f(k, i)C_{ki}^s \langle \nabla_{\hat{u}_s} \hat{u}_j, \hat{u}_l \rangle + \hat{u}_k \langle \nabla_{\hat{u}_i} \hat{u}_j, \hat{u}_l \rangle, \quad i \geq j. \end{aligned} \quad (3.8)$$

Proof. From (3.4) we know that $f_i^a(0) = \delta_i^a$ and that $\frac{\partial}{\partial x^k}|_o = \hat{u}_k|_o = u_k$. Thus, by (3.5)

$$\Gamma_{ij}^l(o) = \langle \Gamma_{ij}^s \frac{\partial}{\partial x^s}, \hat{u}_l \rangle|_o = \langle \frac{\partial f_j^a}{\partial x^i} \hat{u}_a + f_i^b f_j^a \nabla_{\hat{u}_b} \hat{u}_a, \hat{u}_l \rangle|_o.$$

(3.7) is obtained from the above equation. By (3.6) we get

$$\begin{aligned} \left. \frac{\partial \Gamma_{ij}^l}{\partial x^k} \right|_o &= -\Gamma_{ij}^s \Gamma_{ks}^l + f(k, j)C_{kj}^a \langle \nabla_{\hat{u}_i} \hat{u}_a, \hat{u}_l \rangle \\ &\quad + f(k, i)C_{ki}^a \langle \nabla_{\hat{u}_a} \hat{u}_j, \hat{u}_l \rangle + \langle \nabla_{\hat{u}_k} \nabla_{\hat{u}_i} \hat{u}_j, \hat{u}_l \rangle, \quad i \geq j. \end{aligned}$$

And we know that at the origin

$$\begin{aligned} \langle \nabla_{\hat{u}_k} \nabla_{\hat{u}_i} \hat{u}_j, \hat{u}_l \rangle &= \hat{u}_k \langle \nabla_{\hat{u}_i} \hat{u}_j, \hat{u}_l \rangle - \langle \nabla_{\hat{u}_i} \hat{u}_j, \nabla_{\hat{u}_k} \hat{u}_l \rangle, \\ f(k, i)C_{ki}^a \langle \nabla_{\hat{u}_a} \hat{u}_j, \hat{u}_l \rangle &= f(k, i)C_{ki}^s \langle \nabla_{\hat{u}_s} \hat{u}_j, \hat{u}_l \rangle, \end{aligned}$$

and

$$\langle \nabla_{\hat{u}_i} \hat{u}_j, \nabla_{\hat{u}_k} \hat{u}_l \rangle = \Gamma_{ij}^s \langle \nabla_{\hat{u}_k} \hat{u}_l, \hat{u}_s \rangle, \quad i \geq j.$$

From the above four equations (3.8) is obtained. \square

We will also need the following

Lemma 3.2. For $u_i, u_j, u_k, u_l \in \mathfrak{m}, u_\lambda \in \mathfrak{h}$, we have

$$\langle \nabla_{\hat{u}_i} \hat{u}_j, \hat{u}_l \rangle|_o = -\frac{1}{2}(C_{ij}^l + C_{il}^j + C_{jl}^i), \quad (3.9)$$

$$\langle \nabla_{\hat{u}_i} \hat{u}_\lambda, \hat{u}_j \rangle|_o = \langle [u_j, u_\lambda]_{\mathfrak{m}}, u_i \rangle = C_{j\lambda}^i, \quad (3.10)$$

$$\hat{u}_k \langle \nabla_{\hat{u}_i} \hat{u}_j, \hat{u}_l \rangle|_o = \frac{1}{2} \left(C_{ka}^l C_{ij}^a + C_{ka}^j C_{il}^a + C_{ka}^i C_{jl}^a + C_{ij}^s C_{kl}^t \delta_{st} + C_{il}^s C_{kj}^t \delta_{st} + C_{jl}^s C_{ki}^t \delta_{st} \right), \quad (3.11)$$

where $[v_i, v_j]_{\mathfrak{m}}$ denotes the projection of $[v_i, v_j]$ to \mathfrak{m} .

Proof. First, (3.9) is an alternative formulation of (3.3) at the origin in terms of the structure constants. Using the invariance of adu_λ , (3.10) can also be deduced from (3.3). Finally, by (3.3) we have

$$\hat{u}_k \langle \nabla_{\hat{u}_i} \hat{u}_j, \hat{u}_l \rangle = -\frac{1}{2} \hat{u}_k \left(\langle [u_i, u_j], \hat{u}_l \rangle + \langle [u_i, u_l], \hat{u}_j \rangle + \langle [u_j, u_l], \hat{u}_i \rangle \right).$$

Considering the value at the origin o and taking into account (3.2) and (3.1), we can deduce from the above equation that

$$\begin{aligned} \hat{u}_k \langle \nabla_{\hat{u}_i} \hat{u}_j, \hat{u}_l \rangle|_o &= \frac{1}{2} \left(\langle [u_k, [u_i, u_j]]_{\mathfrak{m}}, u_l \rangle + \langle [u_k, [u_i, u_l]]_{\mathfrak{m}}, u_j \rangle \right. \\ &\quad + \langle [u_k, [u_j, u_l]]_{\mathfrak{m}}, u_i \rangle + \langle [u_i, u_j]_{\mathfrak{m}}, [u_k, u_l]_{\mathfrak{m}} \rangle \\ &\quad \left. + \langle [u_i, u_l]_{\mathfrak{m}}, [u_k, u_j]_{\mathfrak{m}} \rangle + \langle [u_j, u_l]_{\mathfrak{m}}, [u_k, u_i]_{\mathfrak{m}} \rangle \right), \end{aligned}$$

from which we get (3.11). \square

By the above two lemmas, at the origin o we have

$$\Gamma_{ni}^j - \Gamma_{nj}^i = \langle \nabla_{\hat{u}_n} \hat{u}_i, \hat{u}_j \rangle - \langle \nabla_{\hat{u}_n} \hat{u}_j, \hat{u}_i \rangle = C_{ji}^n.$$

4. RICCI-QUADRATIC HOMOGENEOUS RANDERS SPACES

In this section we will recall some basic notations about Randers spaces. Let

$$F = \alpha + \beta = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i$$

be a Randers metric. Let $\nabla\beta = b_{i|j}y^i dx^j$ denote the covariant derivative of β with respect to α .

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad s_j := b^i s_{ij}, \quad t_j := s_m s^m{}_j.$$

We use a_{ij} to raise and lower the indices of tensors defined by b_i and $b_{i|j}$. The index “0” means the contraction with y^i . For example, $s_0 = s_i y^i$ and $r_{00} = r_{ij} y^i y^j$, etc. For Ricci-quadratic metrics on Randers spaces, we have the following

Theorem 4.1. [9] *Let $F = \alpha + \beta$ be a Randers metric on an n -dimensional manifold. Then it is Ricci-quadratic if and only if*

$$r_{00} + 2s_0\beta = 2\tilde{c}(\alpha^2 - \beta^2), \quad (4.12)$$

$$s^k{}_{0|k} = (n-1)A_0, \quad (4.13)$$

where $\tilde{c} = \tilde{c}(x)$ is a scalar function and $A_k := 2\tilde{c}s_k + \tilde{c}^2 b_k + t_k + \frac{1}{2}\tilde{c}_k$, here $\tilde{c}_k = \frac{\partial \tilde{c}}{\partial x^k}$.

Now we consider homogeneous Randers spaces. Let $(G/H, \alpha)$ and \mathfrak{m} be as above. If W is a G -invariant vector field on G/H , then the restriction of W to $T_o(G/H)$ must be fixed by the isotropy action of H . Under the identification of $T_o(G/H)$ with \mathfrak{m} , W corresponds to a vector $w \in \mathfrak{m}$ which is fixed by $Ad(H)$. On the other hand, if $w \in \mathfrak{m}$ is fixed by $Ad(H)$, then we can define a vector field W on G/H by $W|_{gH} = \frac{d}{dt}(g \exp(tw)H)|_{t=0}$. Therefore, G -invariant vector fields on G/H are one-to-one corresponding to vectors in \mathfrak{m} fixed by $Ad(H)$. Note that a Randers space $F = \alpha + \beta$ is G -invariant if and only if α and β are both invariant under G . Through α , β corresponds to a vector field \tilde{U} which is invariant under G and satisfying $\alpha(\tilde{U}) < 1$ everywhere. This implies that there is a one-to-one correspondence between the invariant Randers metrics on G/H with the underlying Riemannian metric and the set

$$V = \{u \in \mathfrak{m} | Ad(h)u = u, \langle u, u \rangle < 1, \quad \forall h \in H\}.$$

Also note that in this case the length c of β (or \tilde{U}) is constant.

Let $(G/H, F)$ be a homogeneous Randers space and $(U, (x^1, \dots, x^n))$ be a local coordinate system as in Section 2. We suppose the vector field \tilde{U} which corresponds to the invariant 1-form β corresponds to $u = cu_n$ ($c < 1$) under the Riemannian metric α . Thus

$$\begin{aligned} \tilde{U}|_{gH} &= \frac{d}{dt} g \exp(tu)H|_{t=0} \\ &= \frac{d}{dt} (\exp x^1 u_1 \exp x^2 u_2 \cdots \exp(x^n + ct)u_n)H|_{t=0} \\ &= c \frac{\partial}{\partial x^n} |_{gH}. \end{aligned}$$

Then we have the following (see[6]):

$$\begin{aligned}
b_i &= \beta \left(\frac{\partial}{\partial x^i} \right) = \langle \tilde{U}, \frac{\partial}{\partial x^i} \rangle = c \left\langle \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^i} \right\rangle = c a_{ni}, \\
\frac{\partial b_i}{\partial x^j} &= c \frac{\partial a_{ni}}{\partial x^j} = c (\Gamma_{nj}^k a_{ki} + \Gamma_{ji}^k a_{kn}), \\
b_{i|j} &= \frac{\partial b_i}{\partial x^j} - b_l \Gamma_{ij}^l = c \Gamma_{nj}^k a_{ki}, \\
r_{ij} &= \frac{1}{2} (b_{i|j} + b_{j|i}) = \frac{c}{2} (\Gamma_{nj}^k a_{ki} + \Gamma_{ni}^k a_{kj}), \\
s_{ij} &= \frac{1}{2} (b_{i|j} - b_{j|i}) = \frac{c}{2} (\Gamma_{nj}^k a_{ki} - \Gamma_{ni}^k a_{kj}), \\
s_j &= b^i s_{ij} = a^{il} b_l s_{ij} = c s_{nj}.
\end{aligned} \tag{4.14}$$

Lemma 4.2. *Let $(G/H, F)$ be a homogeneous Randers space and β correspond to u . Then (4.12) implies that*

$$\langle [y, u]_{\mathfrak{m}}, y \rangle = 0, \quad \forall y \in \mathfrak{m}. \tag{4.15}$$

Proof. Considering the value at o , by (4.14), (3.7) and (3.9), we have

$$\begin{aligned}
(r_{00} + 2s_0\beta)|_o &= c\Gamma_{n0}^0 + 2c\frac{c}{2}(\Gamma_{n0}^n - \Gamma_{nn}^0)\langle u, y \rangle \\
&= cC_{0n}^0 + c^2 C_{n0}^n \langle u, y \rangle \\
&= \langle [y, u]_{\mathfrak{m}}, y - \langle u, y \rangle u \rangle.
\end{aligned}$$

It is obvious that

$$2\tilde{c}(\alpha^2 - \beta^2)|_o = 2\tilde{c}(o)(\langle y, y \rangle - \langle u, y \rangle^2).$$

Plugging the above two equations into (4.12), we get that at o

$$\left\langle [y, u]_{\mathfrak{m}}, y - \langle u, y \rangle u \right\rangle = 2\tilde{c}(o) (\langle y, y \rangle - \langle u, y \rangle^2).$$

Setting $y = u$ and taking into account the fact that $\langle u, u \rangle < 1$, we get

$$\tilde{c}(o) = 0.$$

Thus

$$\left\langle [y, u]_{\mathfrak{m}}, y - \langle u, y \rangle u \right\rangle = 0.$$

Replacing y by $y + u$ in the above equation yields

$$\left\langle [y, u]_{\mathfrak{m}}, u + y - \langle u, u + y \rangle u \right\rangle = 0.$$

From the above two equations and the fact that $\langle u, u \rangle < 1$ we deduce that

$$\langle [y, u]_{\mathfrak{m}}, u \rangle = 0, \quad \langle [y, u]_{\mathfrak{m}}, y \rangle = 0.$$

This proves the lemma. \square

Note that in the above lemma we also have

$$\begin{aligned}
C_{ni}^j + C_{nj}^i &= 0 = C_{ni}^n, \\
s_i(o) &= c s_{ni} = \frac{c^2}{2} (\Gamma_{ni}^n - \Gamma_{nn}^i) = \frac{1}{2} \langle [u, u_i]_{\mathfrak{m}}, u \rangle = 0, \\
t_i(o) &= s_m s^m_i = 0.
\end{aligned} \tag{4.16}$$

5. PROOF OF THE MAIN THEOREM

Before the proof, we still need to perform some complicated computation.
First we have

$$\begin{aligned}\frac{\partial b_{i|j}}{\partial x^k} &= \frac{\partial \Gamma_{nj}^l}{\partial x^k} + \Gamma_{nj}^s (\Gamma_{ks}^l + \Gamma_{kl}^t \delta_{ts}) \\ &= f(k, l) C_{kl}^s \Gamma_{nj}^t \delta_{ts} + f(k, j) C_{kj}^a \langle \nabla_{\hat{u}_n} \hat{u}_a, \hat{u}_l \rangle \\ &\quad + C_{kn}^s \langle \nabla_{\hat{u}_s} \hat{u}_j, \hat{u}_l \rangle + \hat{u}_k \langle \nabla_{\hat{u}_n} \hat{u}_j, \hat{u}_l \rangle.\end{aligned}\tag{5.17}$$

Lemma 5.1. *Let $(G/H, F)$ be a homogeneous Randers space and β correspond to u which satisfies (4.15). Then*

$$\frac{\partial b_{0|0}}{\partial x^0} = 0.\tag{5.18}$$

Proof. By the Lemma 3.2 we have the following computations

$$\begin{aligned}\frac{\partial b_{0|0}}{\partial x^0} &= f(i, j) C_{ij}^s y^i y^j \Gamma_{n0}^t \delta_{ts} + f(i, j) C_{ij}^s y^i y^j \langle \nabla_{\hat{u}_n} \hat{u}_s, \hat{u}_0 \rangle + f(i, j) C_{ij}^\lambda y^i y^j \langle \nabla_{\hat{u}_n} \hat{u}_\lambda, \hat{u}_0 \rangle \\ &\quad + C_{0n}^s \langle \nabla_{\hat{u}_s} \hat{u}_0, \hat{u}_0 \rangle + \hat{u}_0 \langle \nabla_{\hat{u}_n} \hat{u}_0, \hat{u}_0 \rangle \\ &= f(i, j) C_{ij}^s y^i y^j (\langle \nabla_{\hat{u}_n} \hat{u}_0, \hat{u}_s \rangle + \langle \nabla_{\hat{u}_n} \hat{u}_s, \hat{u}_0 \rangle) + f(0, 0) C_{00}^\lambda y^i y^j C_{\lambda n}^0 - C_{0n}^s C_{s0}^0 + C_{0a}^0 C_{na}^a \\ &= f(i, j) C_{ij}^s y^i y^j (C_{n0}^s + C_{ns}^0) + f(i, j) C_{ij}^\lambda y^i y^j C_{\lambda n}^0 + C_{0\lambda}^0 C_{n0}^\lambda \\ &= f(i, j) C_{ij}^\lambda y^i y^j C_{\lambda n}^0 \\ &= 0.\end{aligned}\tag{5.19}$$

In above we have used the fact $C_{n\lambda}^i = 0$. \square

Further, we also have the following

Lemma 5.2. *Let $(G/H, F)$ be a homogeneous Randers space and β correspond to u which satisfies (4.15). Then*

$$\frac{\partial s_{k0}}{\partial x^i} |_o = \frac{c}{2} (f(i, k) C_{ik}^s C_{s0}^n + f(i, 0) C_{i0}^s C_{ks}^n + C_{0k}^s C_{is}^n).\tag{5.20}$$

Proof. By (5.17), we get

$$\begin{aligned}\frac{2}{c} \frac{\partial s_{k0}}{\partial x^i} |_o &= \frac{\partial b_{k|0}}{\partial x^i} - \frac{\partial b_{0|k}}{\partial x^i} \\ &= f(i, k) C_{ik}^s \Gamma_{n0}^t \delta_{st} + f(i, 0) C_{i0}^a \langle \nabla_{\hat{u}_n} \hat{u}_a, \hat{u}_k \rangle + C_{in}^s \langle \nabla_{\hat{u}_s} \hat{u}_0, \hat{u}_k \rangle \\ &\quad - (f(i, 0) C_{i0}^s \Gamma_{nk}^t \delta_{st} + f(i, k) C_{ik}^a \langle \nabla_{\hat{u}_n} \hat{u}_a, \hat{u}_0 \rangle + C_{in}^s \langle \nabla_{\hat{u}_s} \hat{u}_k, \hat{u}_0 \rangle) \\ &\quad + \hat{u}_i \langle \nabla_{\hat{u}_n} \hat{u}_0, \hat{u}_k \rangle - \hat{u}_i \langle \nabla_{\hat{u}_n} \hat{u}_k, \hat{u}_0 \rangle \\ &= f(i, 0) (C_{i0}^a \langle \nabla_{\hat{u}_n} \hat{u}_a, \hat{u}_k \rangle - C_{i0}^s \Gamma_{nk}^t \delta_{st}) - f(i, k) (C_{ik}^a \langle \nabla_{\hat{u}_n} \hat{u}_a, \hat{u}_0 \rangle - C_{ik}^s \Gamma_{n0}^t \delta_{st}) \\ &\quad + C_{in}^s (\langle \nabla_{\hat{u}_s} \hat{u}_0, \hat{u}_k \rangle - \langle \nabla_{\hat{u}_s} \hat{u}_k, \hat{u}_0 \rangle) + \hat{u}_i \langle \nabla_{\hat{u}_n} \hat{u}_0, \hat{u}_k \rangle - \hat{u}_i \langle \nabla_{\hat{u}_n} \hat{u}_k, \hat{u}_0 \rangle\end{aligned}$$

By (3.7), (3.9), we have

$$\langle \nabla_{\hat{u}_s} \hat{u}_0, \hat{u}_k \rangle - \langle \nabla_{\hat{u}_s} \hat{u}_k, \hat{u}_0 \rangle = C_{k0}^s.$$

Then by (3.10), we get

$$\begin{aligned}C_{i0}^a \langle \nabla_{\hat{u}_n} \hat{u}_a, \hat{u}_k \rangle - C_{i0}^s \Gamma_{nk}^t \delta_{st} &= C_{i0}^s (\langle \nabla_{\hat{u}_n} \hat{u}_s, \hat{u}_k \rangle - \langle \nabla_{\hat{u}_n} \hat{u}_k, \hat{u}_s \rangle) + C_{i0}^\lambda \langle \nabla_{\hat{u}_n} \hat{u}_\lambda, \hat{u}_k \rangle \\ &= C_{i0}^s C_{ks}^n.\end{aligned}$$

From (3.11) we easily get

$$\hat{u}_i \langle \nabla_{\hat{u}_n} \hat{u}_0, \hat{u}_k \rangle - \hat{u}_i \langle \nabla_{\hat{u}_n} \hat{u}_k, \hat{u}_0 \rangle = C_{is}^m C_{0k}^s + C_{0k}^s C_{in}^t \delta_{st}.$$

Combining the above four equations, we obtain (5.20). \square

Proof of the main theorem. Let G/H be a homogeneous Randers space. Taking the local coordinate system as in Section 3, we have seen that (4.15) holds. In particular, we have $C_{ni}^n = 0$. By (5.20) we have

$$\left. \frac{\partial s_{n0}}{\partial x^0} \right|_o = \frac{c}{2} \left(f(0, n) C_{0n}^s C_{s0}^n + f(0, 0) C_{00}^s C_{ns}^n + C_{0n}^s C_{0s}^n \right) = 0.$$

Differentiating (4.12) and taking into account the fact that $\tilde{c}(o) = 0 = s_0(o)$, we deduce from (4.14) that

$$2\tilde{c}_0(\alpha^2 - \beta^2)|_o = c \frac{\partial b_{0|0}}{\partial x^0} + 2c\beta \frac{\partial s_{n0}}{\partial x^0} = 0.$$

Thus

$$\tilde{c}_0 = 0,$$

which yields

$$A_0(o) = \frac{1}{2} \tilde{c}_0(o) = 0.$$

On the other hand, we have

$$s_{ij}(o) = \frac{c}{2} (\Gamma_{nj}^i - \Gamma_{ni}^j) = \frac{c}{2} C_{ij}^n.$$

Thus

$$\begin{aligned} s^k_{0|k}(o) &= \sum_k s_{k0|k} = \sum_{k,l} \left(\frac{\partial s_{k0}}{\partial x^k} - \Gamma_{kk}^l s_{l0} - \Gamma_{0k}^l s_{kl} \right) \\ &= \frac{c}{2} \sum_{k=1}^n (f(k, 0) - 1) C_{k0}^s C_{ks}^n + \sum_{1 \leq k, l \leq n} \frac{c}{2} \left(C_{l0}^n C_{kl}^k - C_{kl}^n \left(f(0, k) C_{0k}^l - \frac{1}{2} (C_{0k}^l + C_{0l}^k + C_{kl}^0) \right) \right) \\ &= \sum_{1 \leq k, l \leq n} \frac{c}{2} \left(C_{l0}^n C_{kl}^k + \frac{1}{2} C_{kl}^n (C_{0k}^l + C_{0l}^k + C_{kl}^0) \right). \end{aligned}$$

Therefore (4.13) reduces to

$$\sum_{1 \leq k, l \leq n} \frac{c}{2} \left(C_{l0}^n C_{kl}^k + \frac{1}{2} C_{kl}^n (C_{0k}^l + C_{0l}^k + C_{kl}^0) \right) = 0.$$

Set $y = u$ in the above equation. Then by (4.15) we have

$$C_{kl}^n = 0, \quad k, l = 1, \dots, n,$$

i.e.,

$$\langle [u_k, u_l]_{\mathfrak{m}}, u \rangle = 0, \quad k, l = 1, \dots, n. \quad (5.21)$$

It is proved in [6] that a homogeneous Randers space is of the Berwald type if and only if (4.15) and (5.21) hold. This completes the proof. \square

REFERENCES

- [1] D. Bao, S. S. Chern, Z. Shen, An Introduction to Riemann-Finsler Geometry, Springer-Verlag, New York, 2000.
- [2] S. Bacsó and M. Matsumoto, Randers spaces with the h-curvature tensor H dependent on position alone, Publ. Math. Debrecen, 57 (2000) 185-192.
- [3] D. Bao, C. Robles, Z. Shen, Zermelo navigation on Riemannian manifolds, J. Differ. Geom., 66 (3) (2004) 377-435.
- [4] A.L. Besse, Einstein Manifolds. Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1987.
- [5] S.S. Chern, Z. Shen, Riemann-Finsler Geometry, World Scientific Publishers, 2004.
- [6] S. Deng, The S-curvature of homogeneous Randers spaces, Differ. Geom. Appl., 27 (2009) 75-84.
- [7] S. Deng, Z. Hou, Invariant Randers metrics on homogeneous Riemannian manifold, J. Phys. A: Math. Gen., 37 (2004) 4353-4360, Corrigendum, J. Phys. A: Math. Gen., 39 (2006) 5249-5250.
- [8] S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, second ed., Academic Press, 1978.

- [9] B. Li, Z. Shen, Randers metrics of quadratic Riemann curvature, International J. Math., 20 (2009) 1-8.
- [10] B. Najafi, B. Bidabad, A. Tayebi, On R-quadratic Finsler metrics, Iran. J. Sci. Tech. A, 31 (A4) (2007) 439-443.
- [11] G. Randers, On an asymmetric metric in the four-space of general relativity, Physics Rev., 59 (1941), 195-199.
- [12] Z. Shen, R-quadratic Finsler metrics, Publ. Math. Debrecen, 58 (2001) 263-274.
- [13] Z. I. Szabó, Positive definite Berwald spaces, Tensor, N. S., 38 (1981), 25-39.

SCHOOL OF MATHEMATICAL SCIENCES AND LPMC, NANKAI UNIVERSITY, TIANJIN 300071, P R CHINA

COLLEGE OF MATHEMATICAL SCIENCES, TIANJIN NORMAL UNIVERSITY, TIANJIN 300387, P.R. CHINA

E-mail address, Shaoqiang Deng: `dengsq@nankai.edu.cn`

E-mail address, Zhiguang Hu: `nankaitaiji@mail.nankai.edu.cn`